Theory of charge fluctuations and domain relocation times in semiconductor superlattices

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Abstract

Shot noise affects differently the nonlinear electron transport in semiconductor superlattices depending on the strength of the coupling among the superlattice quantum wells. Strongly coupled superlattices can be described by a miniband Boltzmann-Langevin equation from which a stochastic drift-diffusion equation is derived by means of a consistent Chapman-Enskog method. Similarly, shot noise in weakly coupled, highly doped semiconductor superlattices is described by a stochastic discrete drift-diffusion model. The current-voltage characteristics of the corresponding deterministic model consist of a number of stable branches corresponding to electric field profiles displaying two domains separated by a domain wall. If the initial state corresponds to a voltage on the middle of a stable branch and is suddenly switched to a final voltage corresponding to the next branch, the domains relocate after a certain delay time, called relocation time. The possible scalings of this mean relocation time are discussed using bifurcation theory and the classical results for escape of a Brownian particle from a potential well.

1 Introduction

Nonlinear charge transport in semiconductor superlattices has been the object of many theoretical and experimental studies in the past decade [1,2]. Superlattices are artificial spatially periodic structures first proposed by Esaki and Tsu in order to realize a device that exhibits Bloch oscillations [3]. A superlattice (SL) in its simplest form contains a large number of periods, each comprising two layers, which are semiconductors or insulators with different energy gaps, but with similar lattice constants e. g., GaAs and AlAs. These

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SLs are synthesized by molecular-beam epitaxy or related epitaxial growth techniques in the vertical direction. The conduction band edge of an infinitely long ideal SL is modulated so that in the vertical direction it looks like a one-dimensional (1D) crystal, which is formed by a periodic succession of a quantum well (GaAs) and a barrier (AlAs). Typical experiments of vertical charge transport use an undoped or doped SL of finite length placed in the central part of a diode (forming a *p*-*i*-*n* or n^+-n-n^+ structure) with respective contacts at either end of the diode. Depending on the bias condition, the SL configuration, the doping density, the temperature or other control parameters, the current through the SL and the electric field distribution inside the SL display a great variety of nonlinear phenomena such as pattern formation, current self-oscillations, and chaotic behavior [1,2].

To describe and understand nonlinear charge transport in SLs, it is essential to distinguish between *weakly coupled* and *strongly coupled* SLs. Weakly coupled SLs (WCSLs) contain rather thick barriers separating the SL quantum wells, i. e., the barrier width is much larger than the typical electron wavelength inside the barrier. Therefore, a description of the electronic properties of WCSLs can be based on the subband structure of the corresponding isolated quantum well together with resonant tunneling across the barrier of two adjacent wells. In contrast, the quantum wells of strongly coupled SLs (SCSLs) are separated by thin barriers so that the electronic properties of SCSLs can be described in terms of extended states such as Bloch functions. The simplest mathematical models applied to a SL give rise to balance equations involving mesoscopic quantities such as the electric field, the electron density, the drift velocity, etc. The task of deriving these equations from first principles is far from being completed, although reasonable deterministic balance equations have been derived for both SCSLs [4] and for WCSLs [2] in particular limiting cases.

A fundamental difference between WCSLs and SCSLs is that the former are governed by *spatially discrete* balance equations (differential-difference equations), whereas the latter are governed by *spatially continuous* equations (partial differential equations). Both types of equations may have solutions, whose electric field profiles display regions of high electric field coexisting with regions of low electric field. The resulting dynamical behavior is very different for these two types of equations. For SCSLs, which are described by continuous balance equations, the field profile consists of a charge dipole moving with the flow of electrons, which very much resembles the Gunn effect in bulk semiconductors [5]. Under dc voltage bias, this basic motion results in self-sustained oscillations of the current through the SL due to the periodic movement of dipole domains. In contrast, in WCSLs, which are described by discrete balance equations [6,2], electric-field domains (EFDs) are separated by a domain wall, which consists of a charge monopole. The domain wall in a WCSL may move with or opposite to the electron flow, or is pinned, depending on the value of the current [7]. This pinning of the domain wall occurs only in the

discrete models.

In this paper, we are interested in modeling the effects of shot noise in SLs, and then discussing its influence on the motion of EFDs. Shot noise is a consequence of the quantization of the charge [8]. For SCSLs, we shall describe in Section 2 shot-noise effects by using a miniband Boltzmann-Langevin equation, from which we shall derive a stochastic drift-diffusion equation in the hyperbolic limit by means of a consistent Chapman-Enskog method. The resulting stochastic drift-diffusion equation is similar to that describing fluctuations in the Gunn effect, and therefore enhancement of charge fluctuations near the threshold to self-sustained oscillations of the current is to be expected [9].

The discrete balance equations describing electron transport in WCSLs have not been derived consistently from a kinetic theory, and therefore we cannot follow a perturbative treatment as in the case of the SCSLs. Shot-noise effects will therefore be described in Section 3 by Langevin equations resulting from adding appropriate white-noise sources to the discrete equations [10]. An important situation to observe the effects of shot noise is EFD relocation due to voltage switching [11]. The current-voltage characteristics of a WCSL consists of a number of stable branches corresponding to electric field profiles displaying two domains separated by a domain wall. If the initial state corresponds to a voltage on the middle of a stable branch and is suddenly switched to a final voltage corresponding to the next branch, the domains relocate after a certain delay time, called relocation time. The deterministic theory of domain relocation is based on numerical simulations of a nonlinear differential-difference model [12]. One of the main effects of shot noise is to render the relocation time random. If the final voltage after switching is near the limit point that marks the end of the initial stable solution branch, the mean relocation time can be investigated using bifurcation theory. We calculate a possible scaling of this mean relocation time using a stochastic amplitude equation corresponding to the normal form of a saddle-node plus a term due to projected shot noise. For this equation, the classical results for escape of a Brownian particle from a potential well yield the scaling of the mean relocation time. The last Section contains our conclusions.

2 Shot-noise effects in strongly-coupled superlattices

A deterministic simple model of one-dimensional (1D) electron transport in a SCSL consists of the following Boltzmann-Poisson system [4]:

$$\frac{\partial f}{\partial t} + v(k)\frac{\partial f}{\partial x} + \frac{eF}{\hbar}\frac{\partial f}{\partial k} = -\nu_e \left(f - f^{FD}\right) - \nu_i \, \frac{f(x,k,t) - f(x,-k,t)}{2}, \quad (1)$$

$$\varepsilon \, \frac{\partial F}{\partial x} = \frac{e}{l}\,(n - N_D), \quad (2)$$

$$n = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f(x,k,t)dk = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f^{FD}(k;n)dk, \quad (3)$$

$$f^{FD}(k;n) = \frac{m^* k_B T}{\pi\hbar^2} \, \ln\left[1 + \exp\left(\frac{\mu - E(k)}{k_B T}\right)\right]. \quad (4)$$

Here l, ε, f, n, N_D, k_B, T, $-F$, m^* and $-e < 0$ are the SL period, the dielectric constant, the one-particle distribution function, the 2D electron density, the 2D doping density, the Boltzmann constant, the lattice temperature, the electric field, the effective mass of the electron, and the electron charge, respectively. The right side of Eq. (1) contains two collision terms. The first one has BGK (Bhatnagar-Gross-Krook) form, and it represents inelastic energy relaxation towards a 1D effective Fermi-Dirac distribution $f^{FD}(k;n)$ (local equilibrium) with collision frequency ν_e. In Eq. (4), the chemical potential μ depends on n and is found by inverting the exact relation (3). The second collision term accounts for impurity elastic collisions:

$$\begin{aligned} Q_i[f] &= \nu_i \frac{f(x,-k,t) - f(x,k,t)}{2} \\ &= \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} \left[\overline{J}(k',k,x,t) - \overline{J}(k,k',x,t)\right] dk', \end{aligned} \quad (5)$$

$$\overline{J}(k,k',x,t) = \nu_i \, \frac{\pi\Delta|\sin kl|}{2} \delta(E(k) - E(k'))\, f(x,k,t), \quad (6)$$

provided we use the tight-binding miniband dispersion relation, $E(k) = \Delta\,(1-\cos kl)/2$ (Δ is the miniband width), and ignore transversal degrees of freedom. For simplicity, ν_e and ν_i will be fixed constants. The electron velocity in Eq. (1) is $v(k) = E'(k)/\hbar = \Delta l \sin kl/(2\hbar)$.

To include shot-noise effects, we assume that the particle fluxes $\overline{J}(k,k',x,t)$ in the elastic collision term $Q_i[f]$ are replaced by fluctuating fluxes, $J(k,k',x,t) = \overline{J}(k,k',x,t) + \delta J(k,k',x,t)$, where $\langle \delta J \rangle = 0$. Two such fluxes are independent elementary processes unless their arguments are identical. For the same process, the correlations are those of a Poisson process, which yields [8]:

$$\begin{aligned} &\langle \delta J(k_1,k_2,x,t)\delta J(k_1',k_2',x',t')\rangle = \\ &\quad \frac{(2\pi)^2}{Al}\, \delta(k_1 - k_1')\delta(k_2 - k_2')\delta(x - x')\delta(t - t')\, \overline{J}(k_1,k_2,x,t), \end{aligned} \quad (7)$$

in which A is the area of the SL cross section. Inserting the fluctuating terms in Eq. (1), we obtain a Boltzmann-Langevin equation (BLE) whose right side contains an additional term $\xi(x,k,t)$, with zero mean and correlation

$$\langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')\, G(k,k',x,t), \tag{8}$$

$$G(k,k',x,t) = \frac{\pi\nu_i}{A}\, [f(x,k,t)+f(x,-k,t)]\, [\delta(k-k')-\delta(k+k')]. \tag{9}$$

We shall now derive a reduced balance equation for the electric field. In order to do this, we shall assume that the electric field contribution in the BLE is comparable to the deterministic collision terms and that these terms dominate the other three. This is the so-called *hyperbolic limit*, in which the ratio of $\partial f/\partial t$ or $v(k)\partial f/\partial x$ to $(eF/\hbar)\,\partial f/\partial k$ is of order $\epsilon \ll 1$ [4]. Let v_M and F_M be electron velocity and field scales typical of the macroscopic phenomena described by the sought balance equation; for example, let them be the positive values at which the (zeroth order) drift velocity reaches its maximum. In the hyperbolic limit, the time t_0 it takes an electron with speed v_M to traverse a distance $x_0 = \varepsilon F_M l/(eN_D)$, over which the field variation is of order F_M, is much longer than the mean free time between collisions, $\nu_e^{-1} \sim \hbar/(eF_M l) = t_1$. We therefore define $\epsilon = t_1/t_0 = \hbar v_M N_D/(\varepsilon F_M^2 l^2)$ and formally multiply the fluctuating term ξ and the two first terms on the left side of (1) by ϵ. After obtaining the number of desired terms, we set $\epsilon = 1$. The solution of Eq. (1) for $\epsilon = 0$ is straightforwardly calculated in terms of its Fourier coefficients as

$$f^{(0)}(k;n) = \sum_{j=-\infty}^{\infty} f_j^{(0)} e^{ijkl} \tag{10}$$

$$f_j^{(0)} = \frac{1 - ij\varphi/\tau_e}{1+j^2\varphi^2} f_j^{FD}, \tag{11}$$

$$\varphi = \frac{F}{F_M}, \quad F_M = \frac{\hbar\sqrt{\nu_e(\nu_e+\nu_i)}}{el}, \quad \tau_e = \sqrt{1+\frac{\nu_i}{\nu_e}}. \tag{12}$$

Note that Eq. (3) implies $f_0^{(0)} = f_0^{FD} = n$.

Once $f^{(0)}$ is known as a function of the unknown electron density and electric field, we make the Chapman-Enskog ansatz [4]:

$$f(x,k,t;\epsilon) = f^{(0)}(k;n) + \sum_{m=1}^{\infty} f^{(m)}(k;n)\,\epsilon^m, \tag{13}$$

$$\frac{\partial n}{\partial t} = \sum_{m=0}^{\infty} N^{(m)}(n)\,\epsilon^m. \tag{14}$$

The coefficients $f^{(m)}(k;n)$ depend on the 'slow variables' x and t only through

their dependence on the electron density and the electric field (which is itself a functional of n). The electron density obeys a reduced evolution equation (14) in which the functionals $N^{(m)}(n)$ are chosen so that the $f^{(m)}(k;n)$ are bounded and $2\pi/l$-periodic in k. Moreover the condition,

$$\int_{-\pi/l}^{\pi/l} f^{(m)}(k;n)\,dk = 2\pi\, f_0^{(m)}/l = 0, \quad m \geq 1,$$

ensures that $f^{(m)}$, $m \geq 1$, do not contain contributions proportional to the zero-order term $f^{(0)}$. $N^{(m)}(n)$ can be found by integrating (1) over k, using (3), and inserting (13) in the result:

$$N^{(m)}(n) = -l\,\frac{\partial}{\partial x}\int_{-\pi/l}^{\pi/l} v(k) f^{(m)} \frac{dk}{2\pi}. \tag{15}$$

Then integration of (2) over x yields a form of Ampère's law:

$$\varepsilon\frac{\partial F}{\partial t} + \frac{e}{2\pi}\sum_{m=0}^{\infty}\epsilon^m \int_{-\pi/l}^{\pi/l} v(k)\, f^{(m)}(k;n)\,dk = J(t), \tag{16}$$

where $J(t)$ is the total current density.

To find the equations for $f^{(m)}$, we insert Eqs. (13) and (14) in (1), and then we equate like powers of ϵ. The result is the following hierarchy of linear nonhomogeneous equations:

$$Lf^{(1)} = -\left(\frac{\partial}{\partial t} + v(k)\frac{\partial}{\partial x}\right) f^{(0)}\bigg|_0 + \xi^{(0)}(x,k,t), \tag{17}$$

$$Lf^{(2)} = -\left(\frac{\partial}{\partial t} + v(k)\frac{\partial}{\partial x}\right) f^{(1)}\bigg|_0 - \frac{\partial}{\partial t} f^{(0)}\bigg|_1 + \xi^{(1)}(x,k,t), \tag{18}$$

and so on. We have defined $Lu(k) \equiv eF\hbar^{-1}du(k)/dk + (\nu_e + \nu_i/2)u(k) + \nu_i u(-k)/2$, and the subscripts 0 and 1 mean that $\partial n/\partial t$ is replaced by $N^{(0)}(n)$ and by $N^{(1)}(n)$, respectively. In Eq. (17), the correlation of the noise $\xi^{(0)}(x,k,t)$ contains only the zeroth order distribution function. The linear equation $Lu = S$ has a bounded $2\pi/l$-periodic solution provided $\int_{-\pi/l}^{\pi/l} S\,dk = 0$. This solvability condition together with Eqs. (17), (18), etc. also yield the previously found $N^{(m)}$ of Eq. (15) and the reduced equation (13).

The solution of Eq. (17) is

$$f^{(1)} = \nu_e^{-1} \sum_{j=-\infty}^{\infty} \frac{\mathrm{Re}S_j^{(1)} + i\tau_e^{-2}\mathrm{Im}S_j^{(1)} - ij\varphi S_j^{(1)}/\tau_e}{1+j^2\varphi^2}\, e^{ijkl}, \tag{19}$$

in which $S_j^{(1)}$ ($S_0^{(1)} = 0$) is the jth Fourier coefficient of the right hand side of Eq. (17). Using Eqs. (10), (11) and (19), we explicitly write two terms in Eq. (16), thereby obtaining (after setting $\epsilon = 1$) the following *stochastic generalized drift-diffusion equation* (SGDDE) for the electric field:

$$\varepsilon \frac{\partial F}{\partial t} + \tilde{v}\left(F, \frac{\partial F}{\partial x}\right) \frac{eN_D}{l} \left(1 + \frac{\varepsilon l}{eN_D}\frac{\partial F}{\partial x}\right) - D\left(F, \frac{\partial F}{\partial x}\right) \varepsilon \frac{\partial^2 F}{\partial x^2}$$

$$= -\delta J(x,t) + A\left(F, \frac{\partial F}{\partial x}\right) J(t), \tag{20}$$

$$A = 1 + \frac{2ev_M F_M^3 \left[F_M^2 - (1+2\tau_e^2)\, F^2\right]}{\varepsilon l(\nu_e+\nu_i)(F_M^2+F^2)^3}\, n\tilde{m}, \tag{21}$$

$$\tilde{v} = v_M V \tilde{m}\left(A - \frac{\Delta B}{2e}\frac{\partial F}{\partial x}\right), \tag{22}$$

$$V(\varphi) = \frac{2\varphi}{1+\varphi^2}, \quad v_M = \frac{\Delta l\, \tilde{I}_1(M)}{4\hbar\tau_e \tilde{I}_0(M)}, \tag{23}$$

$$\tilde{I}_m(s) = \int_{-\pi}^{\pi} \cos(mk)\, \ln\left(1 + e^{s-\delta+\delta\cos k}\right) dk, \tag{24}$$

$$\tilde{m}\left(\frac{n}{N_D}\right) = \frac{\tilde{I}_1(\mu/k_BT)\, \tilde{I}_0(M)}{\tilde{I}_1(M)\tilde{I}_0(\mu/k_BT)}, \tag{25}$$

$$\tilde{m}_2\left(\frac{n}{N_D}\right) = \frac{\tilde{I}_2(\mu/k_BT)\, \tilde{I}_0(M)}{\tilde{I}_1(M)\tilde{I}_0(\mu/k_BT)}, \tag{26}$$

$$D = \frac{\Delta^2 l F_M}{8\hbar e\tau_e\,(F_M^2+F^2)}\left(1 - \frac{4\hbar v_M C}{\Delta l}\right), \tag{27}$$

$$B = \frac{(5F_M^2 - 4F^2)\tilde{m}_2}{(F_M^2+4F^2)^2\tilde{m}} - \frac{4\hbar v_M F_M^2(F_M^2-F^2)(\tau_e+\tau_e^{-1})(n\tilde{m})'}{\Delta l(F_M^2+F^2)^3}, \tag{28}$$

$$C = \frac{\tau_e(F_M^2 - 2F^2)(n\tilde{m}_2)'}{F_M^2+4F^2} + \frac{8\hbar v_M(1+\tau_e^2)[FF_M(n\tilde{m})']^2}{\Delta l(F_M^2+F^2)^2}. \tag{29}$$

Here the electron density is given by the Poisson equation (2), $\delta = \Delta/(2k_BT)$ and g' means dg/dn. The fluctuating current density $\delta J(x,t)$ in Eq. (20) has zero mean and correlation

$$\langle \delta J(x,t)\delta J(x',t')\rangle = \frac{e^2\Delta^2\nu_i l\, \delta(x-x')\delta(t-t')}{8\pi A\hbar^2\nu_e(\nu_e+\nu_i)(1+\varphi^2)^2}\left(n - \frac{4\hbar\tau_e v_M n\tilde{m}_2}{\Delta l(1+4\varphi^2)}\right). \tag{30}$$

The effects of shot noise on the solutions of the deterministic equation are more noticeable near bifurcation points. Thus they should affect more evidently the

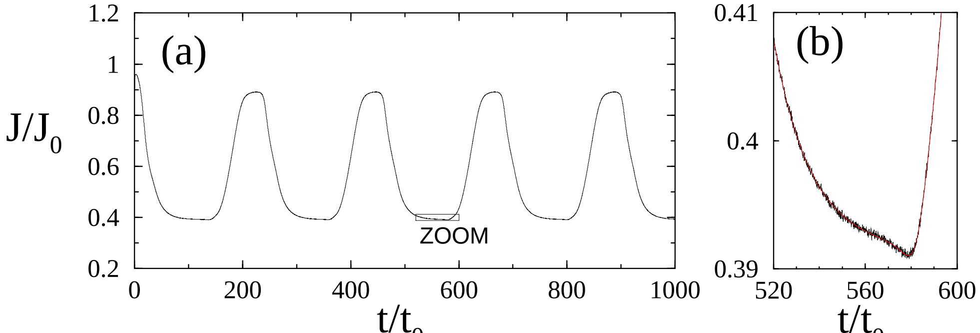

Fig. 1. (a) Self-oscillations of the current density far from bifurcation voltages. (b) Zoom of the time interval marked in (a) showing the effects of the shot noise. Data correspond to a 100-period GaAs/AlAs SL with well and barrier widths of 5.13 and 0.87 nm, respectively, and a 3D doping density of 1.4×10^{17} cm^{-3}, as in [13]. The units of current density and time are $J_0 = 1.9\times10^5$ A/cm^2, $t_0 = 9.7\times10^{-14}$s, respectively, and the voltage is 115% of the threshold voltage above which self-sustained oscillations appear.

onset of the self-sustained oscillations of the current that occur in SCSLs [13]. These stable time-periodic solutions of the deterministic model are due to periodic charge dipole motion from one contact to the other and their frequency agrees with experimental observations [4]. Far from the bifurcation voltages, the effect of shot noise is small, as shown in Fig. 1. However, near the voltage value at which self-oscillations start in the deterministic model, the effects of shot noise are more noticeable. Fig. 2 shows that the shot noise may induce self-oscillations of the current at voltage values below threshold, for which the deterministic model shows relaxation towards the stationary current. A great enhancement in the variance of fluctuations near threshold should also occur, similarly to the effects of noise on the onset of Gunn oscillations in bulk semiconductors [5], cf. Section 7.6 of [9].

3 Shot-noise effects in weakly-coupled superlattices

Deterministic charge transport in weakly coupled SLs is described by discrete balance equations [2]. To this day, no one has derived these equations from quantum kinetic theory, and therefore we cannot study the effects of shot noise on discrete balance equations by perturbative methods similar to those outlined in Section 2. What we can do is to add fluctuating particle fluxes to the discrete balance equations and assume Poissonian statistics for them.

Let us consider a WCSL with $N+1$ barriers and N wells. The zeroth barrier separates the injecting region from the first SL well, whereas the Nth barrier separates the Nth well from the collecting region. Assume that F_i is the average electric field across one SL period, consisting of the ith well and the $(i-1)$th barrier. Similarly, n_i is the 2D electron density at the ith well,

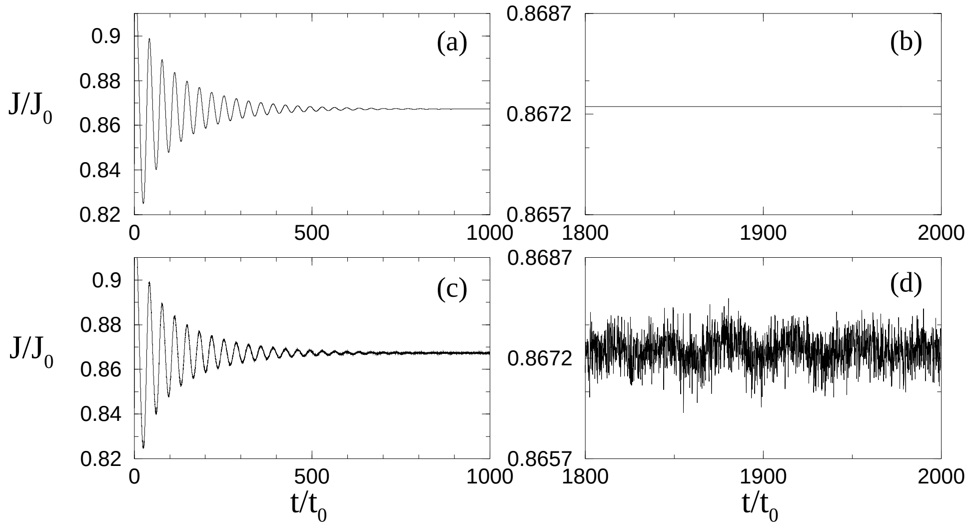

Fig. 2. Self-oscillations of the current density induced by the shot noise for voltages below the threshold voltage. (a) and (b) show the oscillatory relaxation of the current towards its stationary value in the absence of noise. (c) and (d) show the self-sustained oscillations induced when the shot noise is present for the same voltage value. Numerical values are as in Fig. 1, but the voltage is 99.3% of the threshold voltage above which self-sustained oscillations appear.

concentrated in a plane perpendicular to the growth direction inside the ith well. Then, the Poisson equation (averaged over the ith period) and the charge continuity equation are

$$F_i - F_{i-1} = \frac{e}{\varepsilon}\,(n_i - N_D). \tag{31}$$

$$e\,\frac{dn_i}{dt} = J_{i-1\to i} - J_{i\to i+1}. \tag{32}$$

Here $J_{i\to i+1}$ is the tunneling current density across the ith barrier, i. e., from well i to well $i+1$ with $i = 1, \ldots, N$. Assuming that the intersubband scattering times are much smaller than the escape time from a well, which, in turn, is much smaller than the dielectric relaxation time, the main contributions to $J_{i\to i+1}$ are due to sequential resonant tunneling. Electrons in the lowest subband E_1 of the ith well tunnel to one of the excited levels of the $(i+1)$th well, and then immediately relax to the lowest level thereof. In the limit of small γ_ν (broadening of subbands due to scattering), the following expression for the *deteministic* tunneling current, $\overline{J}_{i\to i+1}$, holds [2]:

$$\overline{J}_{i\to i+1} = \frac{e\,v^{(f)}(F_i)}{l}\left\{n_i - \frac{m^* k_B T}{\pi\hbar^2}\,\ln\left[1 + e^{-\frac{eF_i l}{k_B T}}\left(e^{\frac{\pi\hbar^2 n_{i+1}}{m^* k_B T}} - 1\right)\right]\right\}, \tag{33}$$

$$v^{(f)}(F_i) = \sum_{\nu=1}^{n_{max}} \frac{\hbar^3 l}{2m^{*2}}\,\frac{(\gamma_1+\gamma_\nu)T_i(E_1)}{(E_1 - E_\nu + eF_i l)^2 + (\gamma_1+\gamma_\nu)^2}. \tag{34}$$

The *forward drift velocity* $v^{(f)}(F_i)$ is a sum of Lorentzians centered at the resonant field values $F_i = (E_\nu - E_1)/(el)$. $T_i(E_1)$ is proportional to the transmission coefficient through the ith barrier. The tunneling current is a linear function of n_i, but it is a strongly nonlinear function of n_{i+1}. Moreover, $J_{i\to i+1} \sim ev^{(f)}(F_i)n_i/l$, for F_i of the order of the first resonant value or larger. For such values, the resulting tunneling current density has the same shape as assumed in the original *discrete drift model* [6].

Differentiating Eq. (31) with respect to time and inserting the result into Eq. (32), we obtain

$$\varepsilon \frac{dF_i}{dt} + J_{i\to i+1} = J(t), \tag{35}$$

in which the total current density $J(t)$ is the same function for all i. Equation (35) is a discrete version of Ampère's equation. We complete our description by adding the voltage bias condition

$$\frac{1}{(N+1)} \sum_{i=0}^{N} F_i = \frac{V(t)}{(N+1)l}\,, \tag{36}$$

for the known voltage $V(t)$.

Our discrete system of deterministic equations consists of Eq. (31) for $i = 1, \ldots, N$, Eq. (35) for $i = 0, \ldots, N$, and Eq. (36). In total, we have $(2N+2)$ equations for the unknowns $n_1, \ldots, n_N, F_0, \ldots, F_N$, and $J(t)$. We need to specify the constitutive relations $J_{0\to 1}$ (tunneling from the injecting region to the SL), and $J_{N\to N+1}$ (tunneling from the SL to the collecting region). Equation (35) evaluated for $i = 0$ and $i = N$ determines the boundary conditions. As tunneling currents at the boundaries, we use the following phenomenological expressions [12]:

$$\overline{J}_{0\to 1} = \sigma F_0, \quad \overline{J}_{N\to N+1} = \frac{n_N \sigma F_N}{N_D}\,. \tag{37}$$

These conditions are particular cases of the tunneling currents described in [2]. The deterministic model consists of Eqs. (31) and (33) – (37).

In the stochastic model, zero-mean random currents $\delta J_{i\to i+1}(t)$ have to be added to the tunneling current densities $\overline{J}_{i\to i+1}$:

$$J_{i\to i+1} = \overline{J}_{i\to i+1} + \delta J_{i\to i+1}(t). \tag{38}$$

The currents $\delta J_{i\to i+1}$ have correlations [10]

$$
\begin{aligned}
\langle \delta J_{i\to i+1}(t)\,\delta J_{j\to j+1}(t')\rangle &= \frac{e^2 v^{(f)}(F_i)\delta_{ij}\delta(t-t')}{Al} \\
&\times \left\{ n_i + \frac{m^* k_B T}{\pi\hbar^2}\,\ln\left[1+e^{-\frac{eF_i l}{k_B T}}\left(e^{\frac{\pi\hbar^2 n_{i+1}}{m^* k_B T}}-1\right)\right]\right\}, \qquad (39)
\end{aligned}
$$

$$
\langle \delta J_{0\to 1}(t)\,\delta J_{0\to 1}(t')\rangle = \frac{e\sigma\, F_0}{A}\,\delta(t-t'), \qquad (40)
$$

$$
\langle \delta J_{N\to N+1}(t)\delta J_{N\to N+1}(t')\rangle = \frac{e n_N \sigma F_N}{A N_D}\,\delta(t-t'), \qquad (41)
$$

for $i = 1, \ldots, N-1$. The idea behind this form of random tunneling current is that uncorrelated electrons are arriving at the i-th barrier with a distribution function of time intervals between arrival times that is Poissonian [8]. Moreover, the correlation time is of the same order as the tunneling time, so that it is negligible on the longer time scale of dielectric relaxation.

The high-temperature limit of the stochastic model has been numerically solved in [10]. Here we shall study a particular effect of shot noise on the nonlinear dynamics of a WCSL. The current-voltage characteristics of the corresponding deterministic model consist of a number of stable solution branches corresponding to electric field profiles displaying two domains separated by a domain wall. The current-voltage plane corresponds to the usual bifurcation diagram of a norm of a solution (the total current density) versus the bifurcation parameter (the voltage). We should note that two neighboring stable solution branches are typically connected via an intermediate unstable solution branch at two limit points that are saddle-node bifurcations. If the initial state corresponds to a voltage on the middle of a stable branch, V_i, and is suddenly switched to a final voltage corresponding to the next branch, V_f, the domains relocate after a certain delay time, called relocation time [12].

The possible scalings of this mean relocation time can be calculated by means of bifurcation theory and the classical results for escape of a Brownian particle from a potential well. The idea is as follows. Let (V_L, J_L) be the limit point connecting one stable and one unstable branch of deterministic stationary solutions via a saddle-node bifurcation: two solutions exist for $V < V_L$, and they disappear for $V > V_L$. Let us assume that $V_i < V_f < V_L$, and that for $V = V_f$ three stationary solutions coexist with current densities $J_1 < J_2 < J_3$. The solution branch corresponding $J = J_3$ is the same stable branch containing the initial point. This branch coalesces with the branch of unstable solutions corresponding to J_2 at the limit point (V_L, J_L). Provided $(V_f - V_L)$ is of the same order as the noise amplitude in the appropriate dimensionless units, a projection of the noise on the eigenvector corresponding to the zero eigenvalue in the linear stability problem at $V_f = V_L$ appears in the amplitude equation for the saddle-node bifurcation. The latter becomes the following stochastic

amplitude equation:

$$\frac{da}{dt} = \alpha\,(V_f - V_L) + \beta a^2 + \sqrt{2\gamma}\,\xi(t). \tag{42}$$

In this expression, a is proportional to $|J - J_L|$ and $\xi(t)$ is the zero-mean, delta-correlated white noise. The positive parameters α, β, and γ can be calculated by projection methods [14], but their precise form does not matter for the argument we want to make. Note that Eq. (42) has two stationary solutions with $|J - J_L| \propto |V_f - V_L|^{1/2}$ if $\gamma = 0$.

For $V_f < V_L$, Eq. (42) describes the escape of a Brownian particle from a potential well corresponding to the cubic potential $U(a; V_f - V_L) = \alpha\,(V_f - V_L)\,a + \beta a^3/3$. Provided the height of the barrier is large compared to the noise strength, the reciprocal of the mean escape time is proportional to the equilibrium probability density $P = e^{-U/\gamma}/Z$ evaluated at the maximum of the potential. This yields

$$\tau_{reloc} \sim \frac{\pi}{\sqrt{\alpha\beta\,|V_f - V_L|}} \exp\left(\frac{4\,|\alpha(V_f - V_L)|^{3/2}}{3\gamma\sqrt{\beta}}\right), \tag{43}$$

cf. Eq. (8.6.17) in [15]. Thus, there exists a relatively large voltage interval $|V_f - V_L|^{3/2} \gg \gamma\beta^{1/2}\alpha^{-3/2}$, over which the logarithm of the relocation time scales superlinearly with $|V_f - V_L|$. In terms of the current, $\ln(\tau_{reloc}) \propto |J - J_L|^3$, because $a \propto |J - J_L| = O(|V_f - V_L|^{1/2})$.

There is some evidence that the parabolic region near the limit point might be so narrow that the amplitude equation (42) breaks down. For example, Fig. 5 of Ref. [16] shows that the maximum and the limit point of the current-voltage characteristic are extremely close for all branches. In this case, it may very well happen that the region $\gamma^{2/3}\beta^{1/3}/\alpha \ll |V_f - V_L| \ll K$ in which Eq. (42) holds is too narrow. This case might be better approximated by analyzing the effect of shot noise on a spiky limit point, at which the slope of the current-voltage characteristic is discontinuous, and the term proportional to a^2 in (42) is replaced by a piecewise linear function of a. Then a simple calculation of the barrier height yields $\ln(\tau_{reloc}) \propto |V_f - V_L|^2 \propto |J - J_L|^2$ instead of Eq. (43). The numerical solution of the stochastic equations could be used to discriminate between this prediction and Eq. (43). There are early experiments in which $\ln(\tau_{reloc})$ was fitted to a straight line [11], but no theoretical prediction existed at that time and perhaps more careful measurements should be attempted in order to validate the theory.

4 Conclusions

We have modelled the effects of shot noise in strongly and weakly coupled superlattices. For a SCSL, we have proposed a Boltzmann-Langevin equation containing fluctuating terms that represent shot noise. Then we have used a consistent Chapman-Enskog method to derive a stochastic drift-diffusion equation for the electric field. In the case of WCSLs, we have slightly generalized the discrete Langevin equations proposed in [10], and found the scaling for the mean relocation time that electric field domains take to react to a switch in the applied voltage.

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